

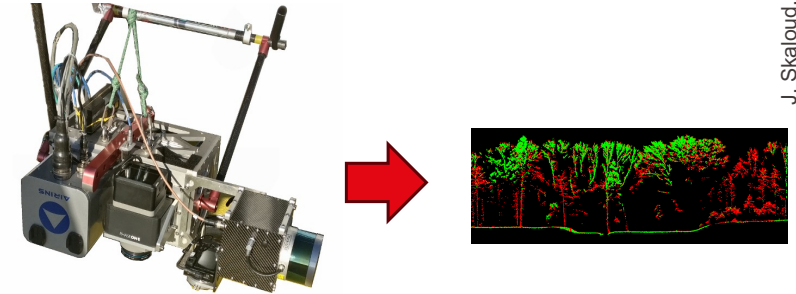


Sensor Orientation – Random Processes

Jan SKALoud

Sensor orientation – main topics

This translates into three rough big areas



1. Fundamentals

- How to characterize sensor noise
- How to transform from the sensed signals to navigation frame?

2. Position, velocity, attitude (navigation)

- How to formulate navigation equation in different frames?
- How to resolve them numerically?

3. Sensor fusion

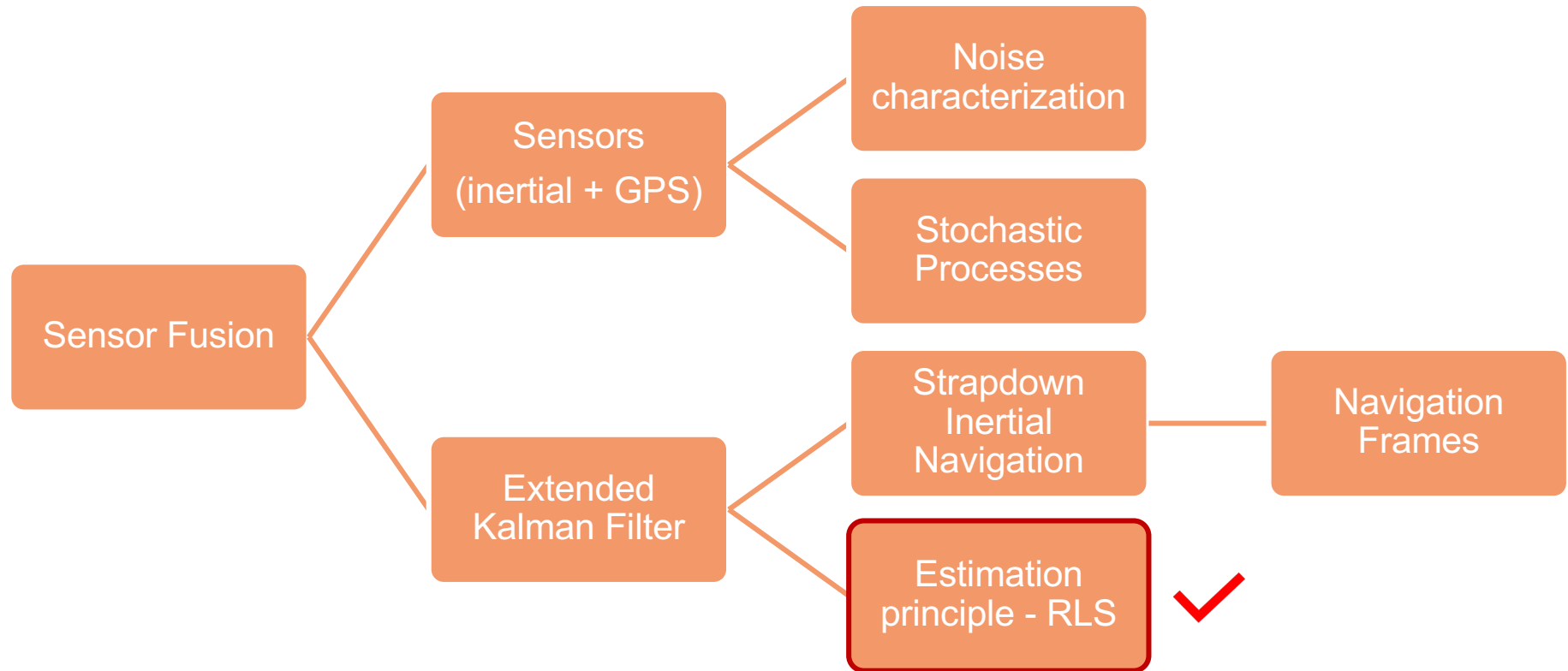
- How to formulate models for sensor fusion?
- How to implement it in optimization and use it for mapping?

You need the frames

You need the navigation quantities and the noise properties

Cockpit view of SO course's topics

How to reach *integrated* sensor orientation?



Similarities RLS vs. Kalman Filter

(2.2.3)

Recursive Least Square (**RLS**)

(2.28-2.30) gain, state,
covariance:

$$\begin{aligned}\mathbf{K}_j &= \mathbf{P}_{j-1} \mathbf{H}_j^T (\mathbf{H}_j \mathbf{P}_{j-1} \mathbf{H}_j^T + \mathbf{R}_j)^{-1} \\ \hat{\mathbf{x}}_j &= \hat{\mathbf{x}}_{j-1} + \mathbf{K}_j (\mathbf{z}_j - \mathbf{H}_j \hat{\mathbf{x}}_{j-1}) \\ \mathbf{P}_j &= (\mathbf{I} - \mathbf{K}_j \mathbf{H}_j) \mathbf{P}_{j-1}\end{aligned}$$

Discrete Kalman Filter (KF)

(2.32-2.34) gain, state,
covariance:

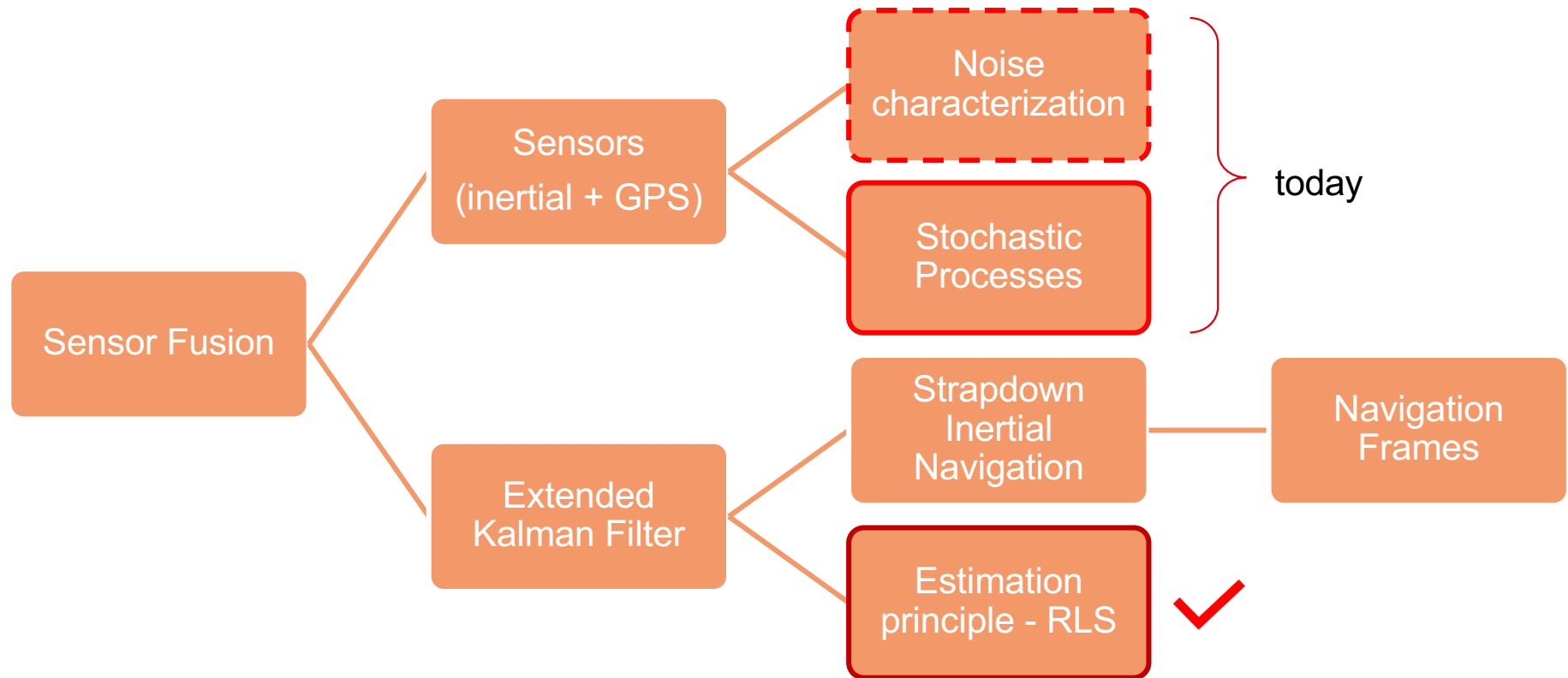
$$\begin{aligned}\mathbf{K}_k &= \tilde{\mathbf{P}}_k \mathbf{H}_k^T (\mathbf{H}_k \tilde{\mathbf{P}}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \\ \hat{\mathbf{x}}_k &= \tilde{\mathbf{x}}_k + \mathbf{K}_k (\mathbf{z}_k - \mathbf{H}_k \tilde{\mathbf{x}}_k) \\ \mathbf{P}_k &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \tilde{\mathbf{P}}_k\end{aligned}$$

Time update (prediction)

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1} &= \Phi_k \hat{\mathbf{x}}_k \\ \tilde{\mathbf{P}}_{k+1} &= \Phi_k \mathbf{P}_k \Phi_k^T + \boxed{\mathbf{Q}_k} \text{ system / sensor noise !}\end{aligned}$$

Cockpit view of SO course's topics

How to reach *integrated* sensor orientation?



Stochastic Processes – agenda (Ch 5)

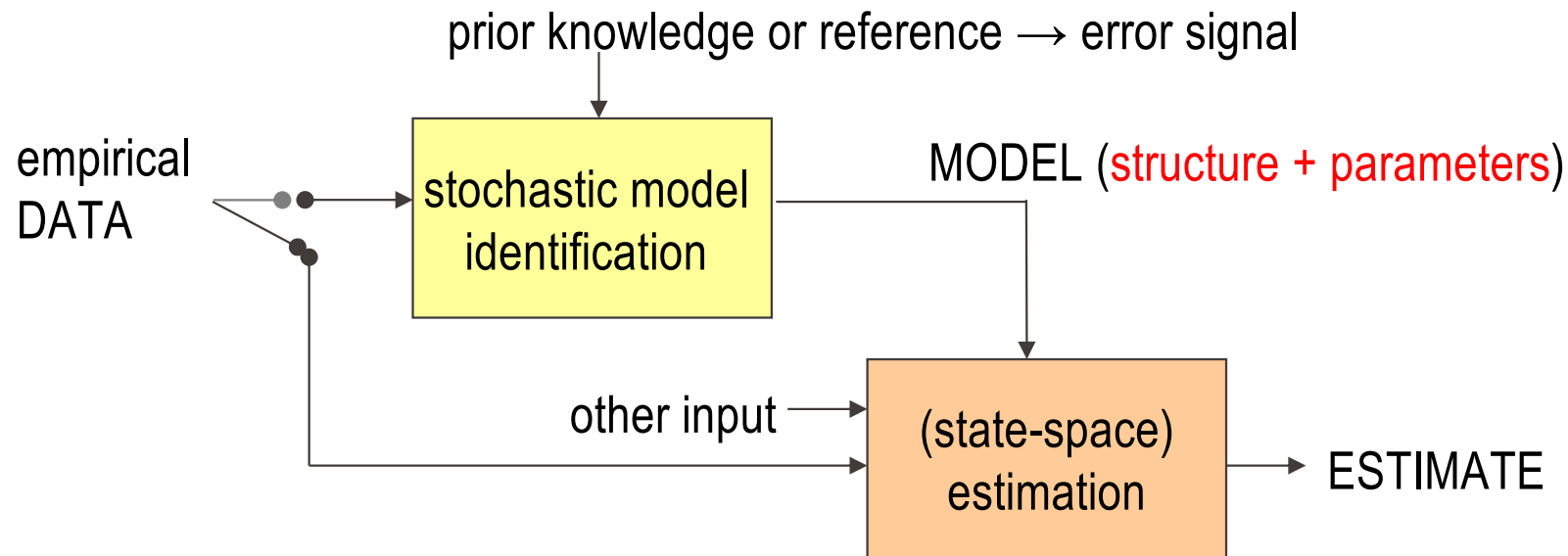
- Introduction
 - Model identification
 - Concept & classification

- Process characteristics
 - Auto-correlation (**AC**)
 - Power Spectral Density (**PSD**)
 - Allan Variance (**AV**) / Wavelet Variance (**WV**)

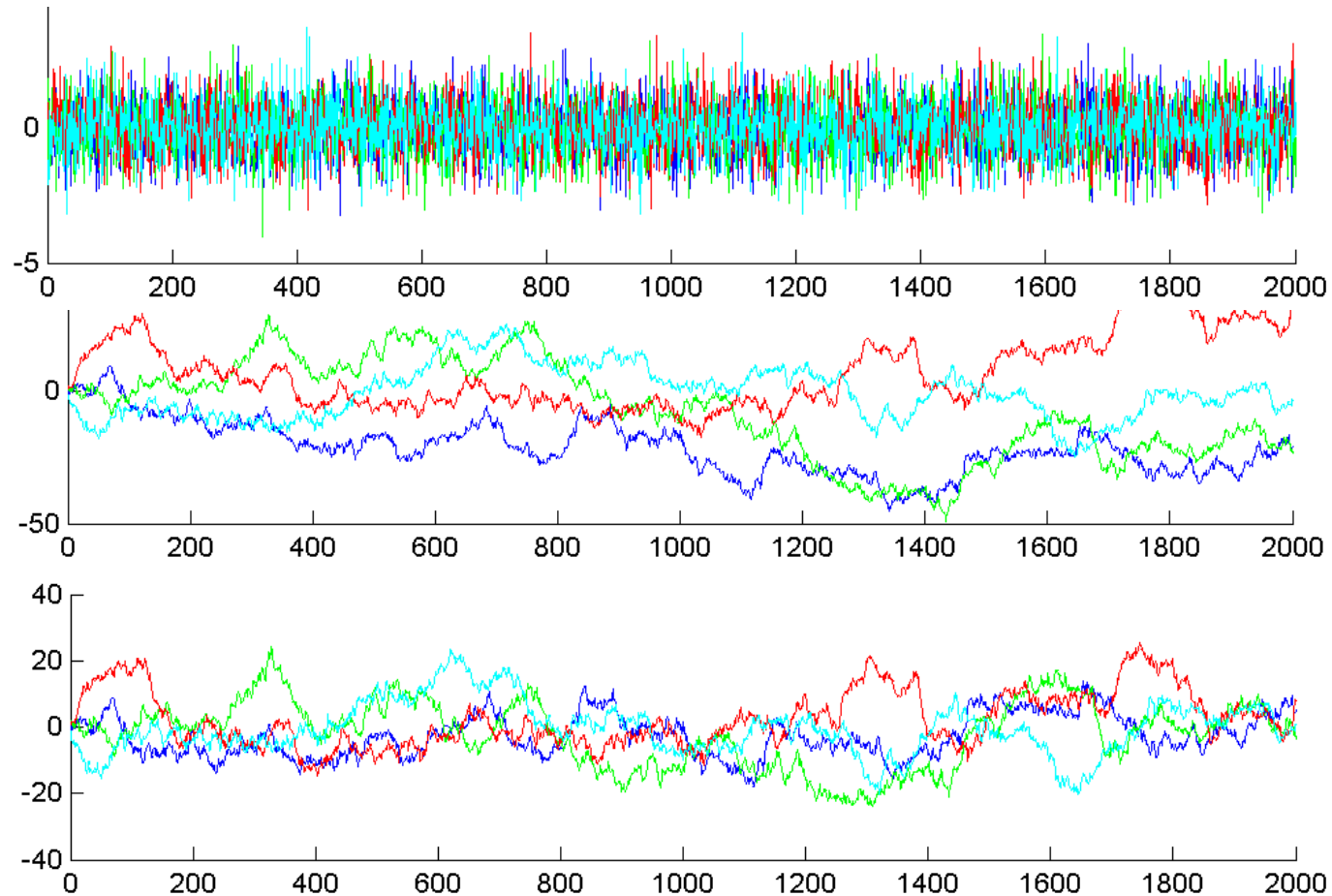
- The very useful processes
 - White Noise
 - Random bias
 - Random walk
 - Auto-regressive white noise (Gauss-Markov)

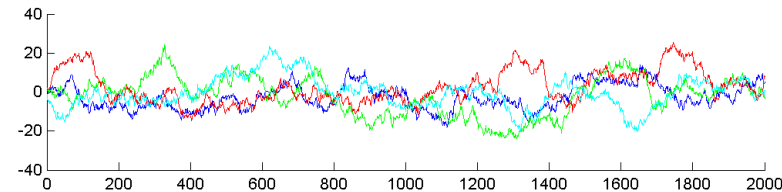
Noise model (process) identification

- Stochastic model
 - Prerequisite for sensor fusion (e.g. Kalman filtering)
 - Two elements needed:
 - structure (type of parameters)
 - Values of parameter
- How to identify?



Random process – an example





Association of random variable(s) with a deterministic parameter (e.g. time)

Stochastic **collection** – values depend on the outcome of an experiment

Time is a **parameter**, but the process is only LOOSELY (stochastic) function of time

Can be either *discrete* or continuous

Stochastic Processes – agenda (Ch 5)

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Connections to random variables

Random variable

Probability density

$$f(x) \quad f(y)$$

Random **process**

Probability density

$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Connections to random variables

Random variable

Probability density

$$f(x) \quad f(y)$$

Expectation

$$E(x) = \mu_X = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(y) = \mu_Y = \int_{-\infty}^{\infty} y f(y) dy$$

Random **process**

Probability density

$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Expectation

$$\mu_X(t) = \int_{-\infty}^{\infty} x_t f(x_t) dx_t$$

Connections to random variables

Random variable

Probability density

$$f(x) \quad f(y)$$

Expectation

$$E(x) = \mu_X = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(y) = \mu_Y = \int_{-\infty}^{\infty} y f(y) dy$$

Covariance

$$\text{cov}(x, y) = E[(x - \mu_X)(y - \mu_Y)]$$

Random process

Probability density

$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Expectation

$$\mu_X(t) = \int_{-\infty}^{\infty} x_t f(x_t) dx_t$$

Joint expectation – autocovariance

$$\text{acov}_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \mu_{X_1})(x_2 - \mu_{X_2}) f(x_1, x_2) dx_1 dx_2$$

Connections to random variables

Random variable

Probability density

$$f(x) \quad f(y)$$

Expectation

$$E(x) = \mu_X = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(y) = \mu_Y = \int_{-\infty}^{\infty} y f(y) dy$$

Covariance

$$\text{cov}(x, y) = E[(x - \mu_X)(y - \mu_Y)]$$

Variance

$$\text{var}(x) = \text{cov}(x, x) = E[(x - \mu_X)^2]$$

Random process

Probability density

$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Expectation

$$\mu_X(t) = \int_{-\infty}^{\infty} x_t f(x_t) dx_t$$

Joint expectation – autocovariance

$$\text{acov}_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \mu_{X_1})(x_2 - \mu_{X_2}) f(x_1, x_2) dx_1 dx_2$$

Variance

$$\text{acov}_X(\tau = 0) = \varphi_{xx}(0) = \sigma_X^2$$

Other characteristics

Random variable

Probability density

$$f(x)$$

Fourier transform of
probability density
(characteristic function)

$$\begin{aligned} g(\omega) &= E[e^{-i\omega x}] \\ &= \int_{-\infty}^{\infty} e^{-i\omega x} f(x) dx \end{aligned}$$

Random process

Probability density

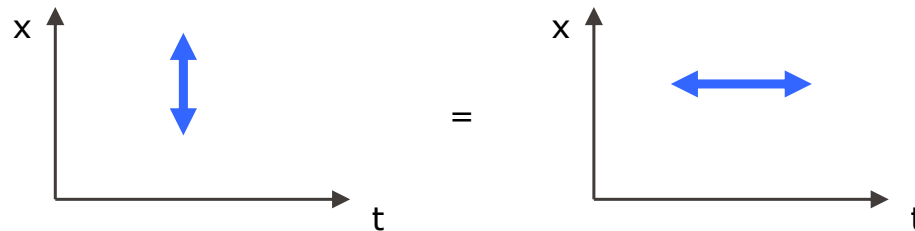
$$f(x_i, t_i), f(x_j, t_j) \longrightarrow f_2(x_i, x_j)$$

Power spectral density – **PSD**
(Fourier transform of
autocovariance function)

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau) e^{-i 2\pi \omega \tau} d\tau$$

Classification of random processes

□ By ergodicity*



$$E(x) = \lim_{t \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt$$

$$E(x^2) = \lim_{t \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x^2(t) dt$$

$$\varphi_{xx}(t_1, t_2) = \lim_{t \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t_1) x(t_2) dt$$

* Process characteristics computed over one long process are expected to be same as those computed from many processes sampled at a specific time

Classification of random processes

By **stationarity**

- Process characteristics are constant in time (t)
- i.e. can be computed from a "sample interval" (rather than absolute time)

$$\varphi_{xx}(\tau) = E(x(t)x(t+\tau))$$

$$\varphi_{xx}(0) = E(x^2)$$

$$\varphi_{xx}(-\tau) = \varphi_{xx}(\tau)$$

Autocorrelation function (AC)

Autocorrelation function:

- Assumption **1**: Underlying process is **stationary**
Gaussian random process
- Assumption **2**: Data (z) are sampled at **constant Δt**

$$m = \frac{1}{N} \sum_{i=1}^N z_i$$

$$\varphi_{zz}(l\Delta t) = \frac{1}{N-l-1} \sum_{i=1}^{N-l} (z_i - m)(z_{i+l} - m)^T; l = 0, 1, \dots, N-2$$

Other time series (or frequency) analysis (later):

- AR, MA, ARMA, ARIMA, ... + estimators

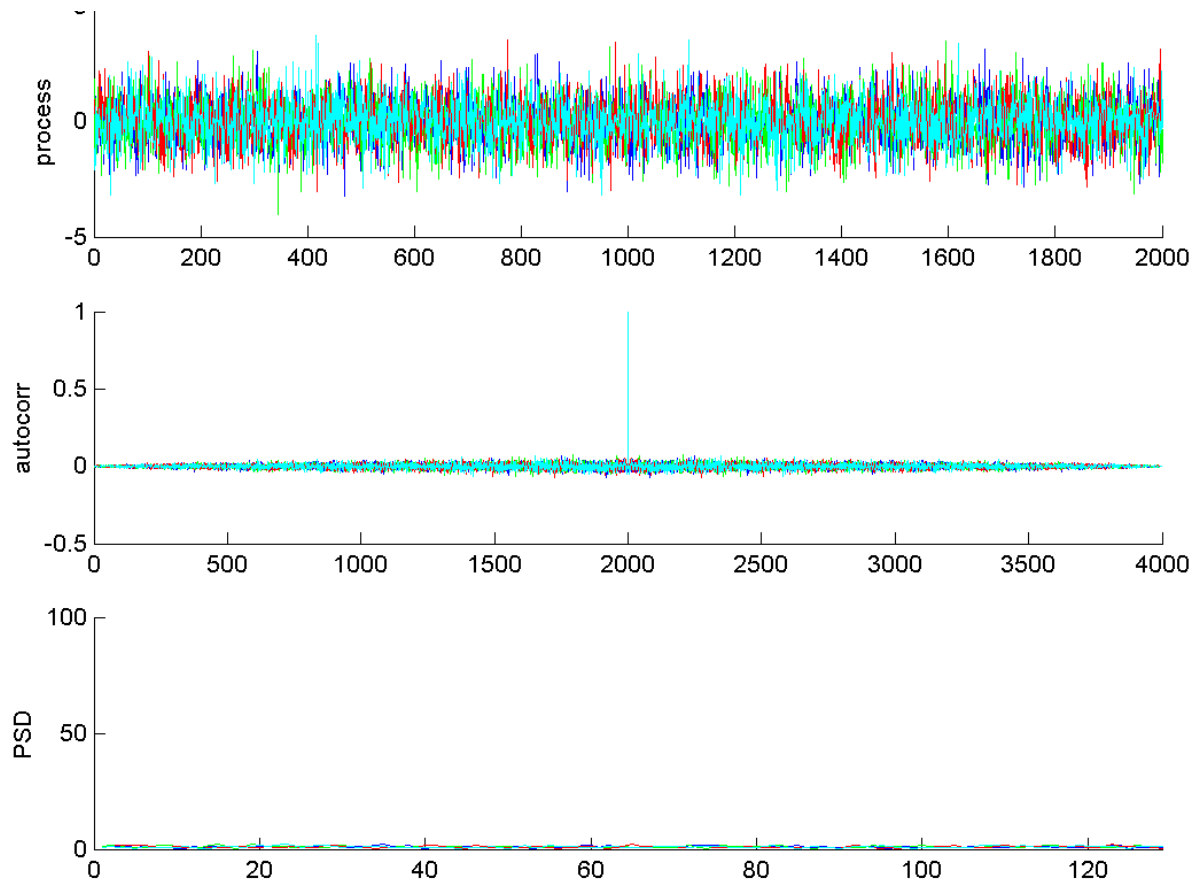
Stochastic Processes – agenda (Ch 5)

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White noise - example



■ Sensor orientation

White noise process

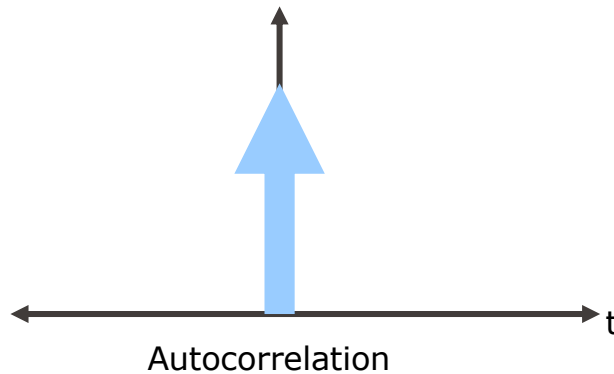
- State propagation:

$$x_k = w_k$$

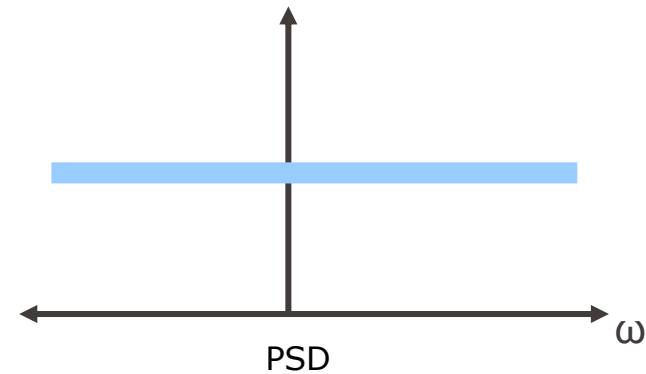
- Differential equation:

theoretically not differentiable

$$\varphi_{xx}(\tau) = \varphi_{xx}(0)\delta(\tau)$$



$$\Phi_{xx}(\omega) = \sigma^2$$



Random bias (constant)

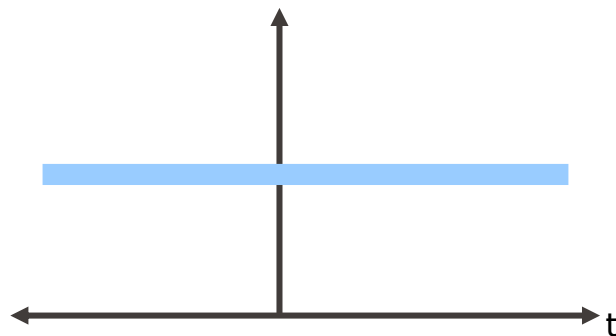
□ State propagation:

$$x_{k+1} = x_k = m$$

□ Differential equation:

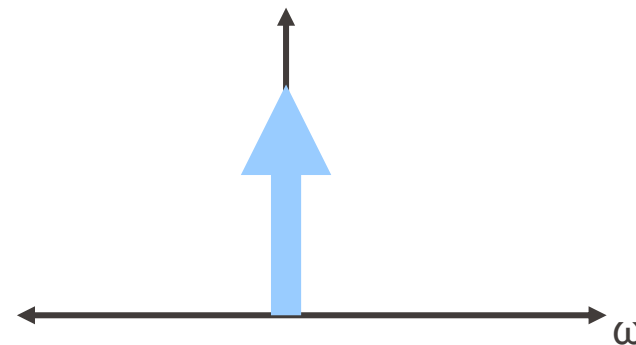
$$\dot{x}(t) = 0$$

$$\varphi_{xx}(\tau) = m^2$$



Autocorrelation

$$\Phi_{xx}(\omega) = 2\pi m^2 \delta(\omega)$$



PSD

Random walk

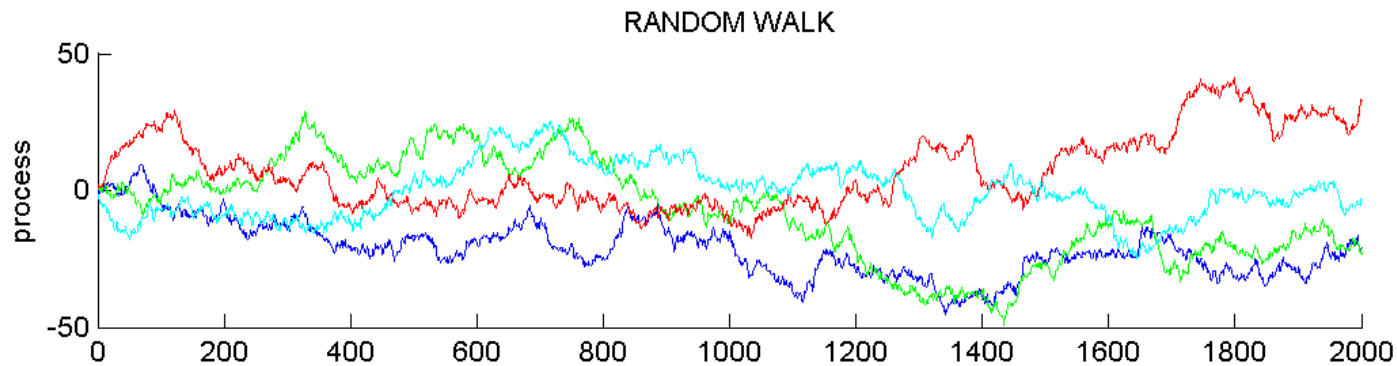
□ Differential equation:

$$\dot{x}(t) = w(t)$$

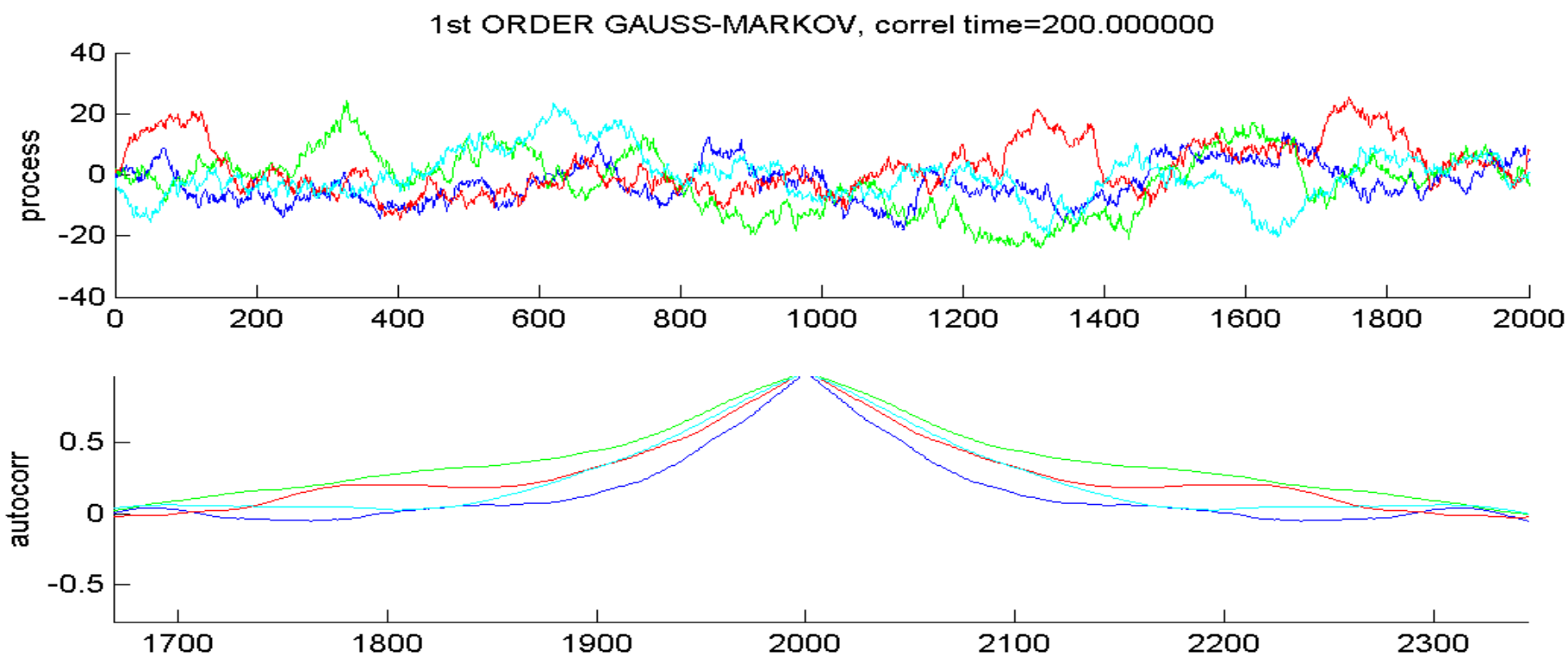
□ State propagation:

$$x_{k+1} = x_k + w_k$$

□ NOT a stationary process!



1st order Gauss-Markov process - example



1st order Gauss-Markov process

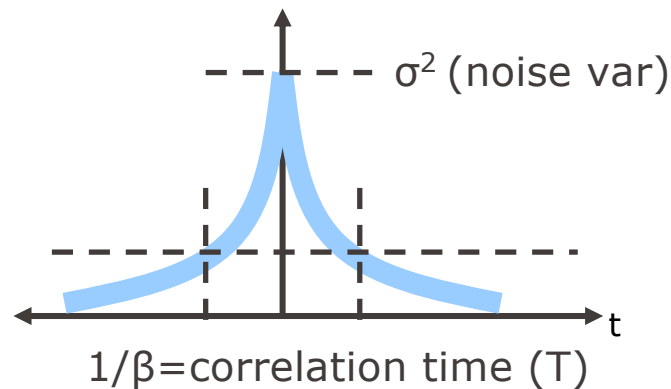
□ Differential equation:

$$\dot{x}(t) = -\beta x(t) + w(t)$$

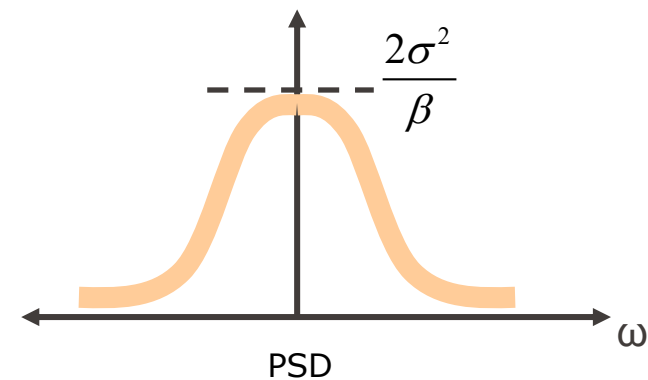
□ State propagation:

$$x_{k+1} = e^{-\beta \Delta t} x_k + w_k$$

$$\varphi_{xx}(\tau) = \sigma^2 e^{-\beta|\tau|}$$



$$\Phi_{xx}(\omega) = \frac{2\beta\sigma^2}{\omega^2 + \beta^2}$$



Other random processes

Random ramp

Quantization noise

Periodic (sinusoid)

Auto-Regressive (AR) process

Moving-Average (MA) process

Auto-Regressive Moving Average (ARMA)

...

Stochastic Processes – agenda (Ch 5)

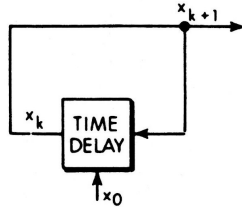
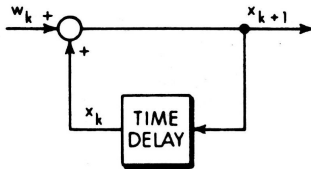
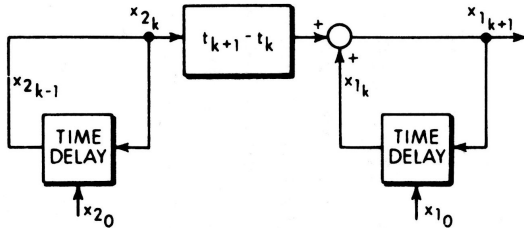
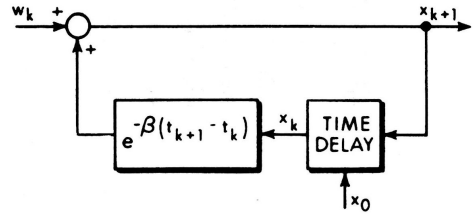
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Error Models – Discrete Case

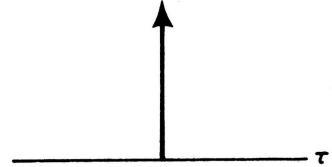
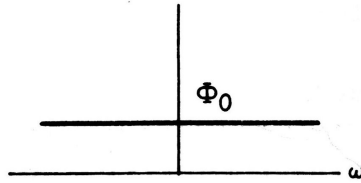
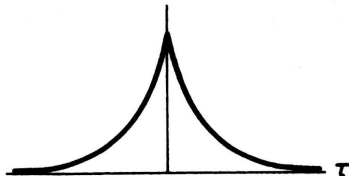
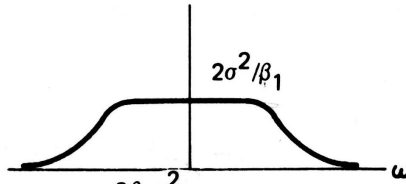
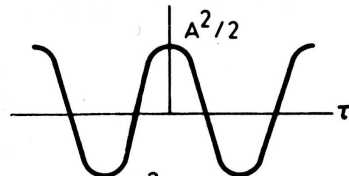
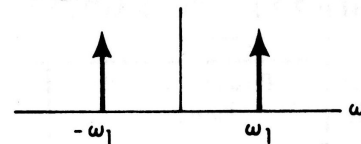
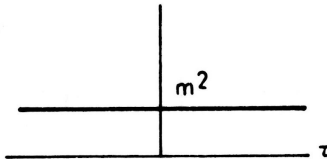
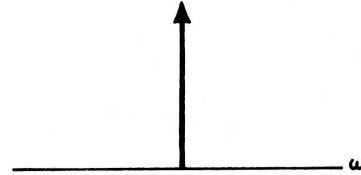
(distributed)

NAME	STATE DIFFERENCE EQUATION	BLOCK DIAGRAM	RELATION of DISCRETE q_k to CONTINUOUS q
RANDOM CONSTANT	$x_{k+1} = x_k$		$q_k = 0$
RANDOM WALK	$x_{k+1} = x_k + w_k$		$q_k = q(t_{k+1} - t_k)$
RANDOM RAMP	$x_{1k+1} = x_{1k} + \Delta t x_{2k}$ $x_{2k+1} = x_{2k}$		$q_k = 0$
EXPONENTIALLY CORRELATED RANDOM VARIABLE	$x_{k+1} = e^{-\beta(t_{k+1} - t_k)} x_k + w_k$		$q_k = \frac{q}{2\beta} \left[1 - e^{-2\beta(t_{k+1} - t_k)} \right]$

After Gelb, A. (1974)

Common Random Processes

(distributed)

PROCESS	AUTOCORRELATION FUNCTION	POWER SPECTRAL DENSITY
WHITE NOISE	 $\phi_{xx}(\tau) = \phi_0 \delta(\tau)$	 $\Phi_{xx} = \Phi_0$
MARKOV PROCESS $m = 0$	 $\phi_{xx}(\tau) = \sigma^2 e^{-\beta_1 \tau }$	 $\Phi_{xx} = \frac{2\beta_1 \sigma^2}{\omega^2 + \beta_1^2}$
SINUSOID	 $\phi_{xx}(\tau) = \frac{A^2}{2} \cos \omega_1 \tau$	 $\Phi_{xx} = \frac{\pi}{2} A^2 [\delta(\omega - \omega_1) + \delta(\omega + \omega_1)]$
RANDOM BIAS	 $\phi_{xx}(\tau) = m^2$	 $\Phi_{xx} = 2\pi m^2 \delta(\omega)$

After Gelb, A. (1974)

Autocorrelation function (AC) - estimation

Autocorrelation function:

- Assumption **1**: Underlying process is **stationary**
Gaussian random process
- Assumption **2**: Data (z) are sampled at **constant Δt**

$$m = \frac{1}{N} \sum_{i=1}^N z_i$$

$$\varphi_{zz}(l\Delta t) = \frac{1}{N-l-1} \sum_{i=1}^{N-l} (z_i - m)(z_{i+l} - m)^T; l = 0, 1, \dots, N-2$$

Other time series (or frequency) analysis (later):

- AR, MA, ARMA, ARIMA, ... + estimators

Autocorrelation function (AC) - estimation

MATLAB

`xcorr(...,SCALEOPT)`

- normalizes the correlation according to SCALEOPT:

1. 'biased' - scales the raw cross-correlation by $1/M$ ($M=\text{vec length}$).
2. 'unbiased' - scales the raw correlation by $1/(M-\text{abs}(\text{lags}))$.
3. 'normalized' or 'coeff' - normalizes the sequence so that the auto-correlations at zero lag are identically 1.0.
4. 'none' - no scaling (this is the default).

Power spectrum (or PSD)

- PSD is the Discrete Fourier Transform (DFT) of AC:

$$\Phi_{xx}(\omega) = \sum_{\tau=-\infty}^{\infty} \varphi_{xx}(\tau) e^{-j\omega\tau}$$

- If we have a signal of length N:

- Sample AC (AC estimate):

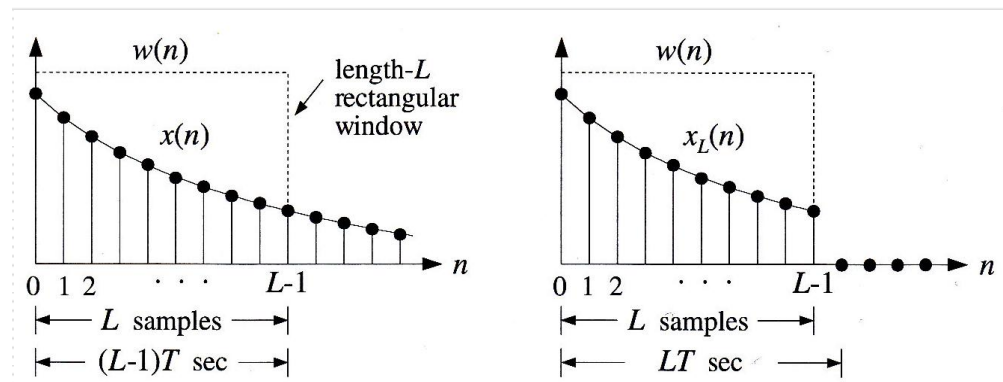
$$\hat{\varphi}_{xx}(\tau) = \frac{1}{N-\tau-1} \sum_{n=0}^{N-1-\tau} x_{n+\tau} x_n; \quad \tau = 0, 1, \dots, N-1$$

- Periodogram spectrum (PSD estimate)

$$\hat{\Phi}_{xx}(\omega) = \sum_{\tau=-\infty}^{\infty} \hat{\varphi}_{xx}(\tau) e^{-j\omega\tau}$$

General note about windowing in DFT

- Time windowing -rectangular window for finite data



$$x(nT) = \varphi_{xx}(\tau)$$

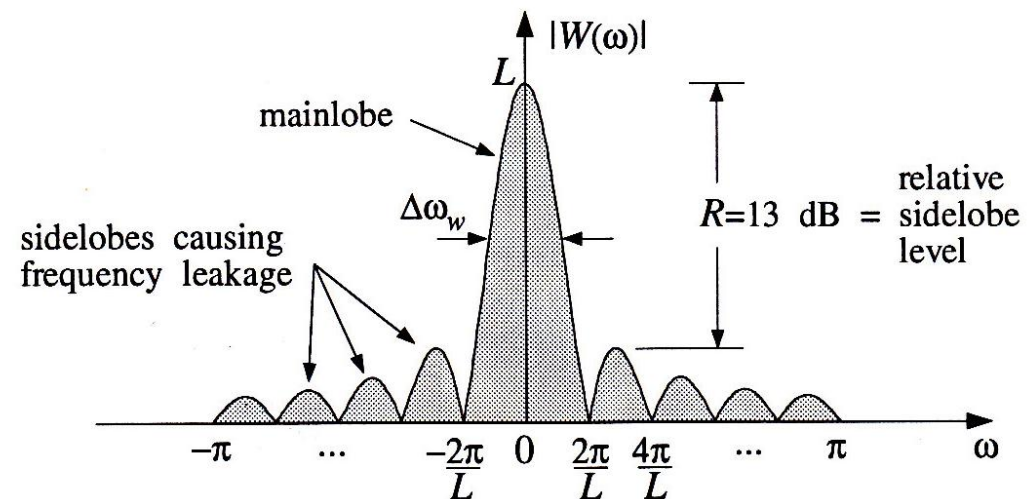
- Original and time-windowed spectrum:

$$\hat{\Phi}(\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT}$$

$$\hat{\Phi}_L(\omega) = \sum_{n=0}^{L-1} x_L(nT) e^{-j\omega nT} = \sum_{n=-\infty}^{\infty} x_L(n) e^{-j\omega n}$$

General note about windowing in DFT

- Magnitude spectrum of rectangular window:



$$|W(\omega)| = \left| \frac{\sin(\omega L / 2)}{\sin(\omega / 2)} \right|$$

General note about windowing in DFT

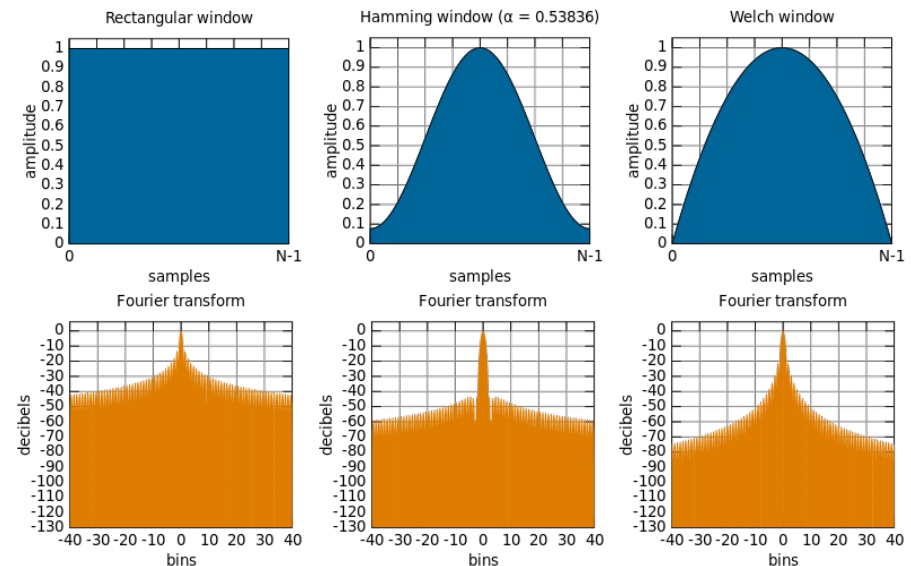
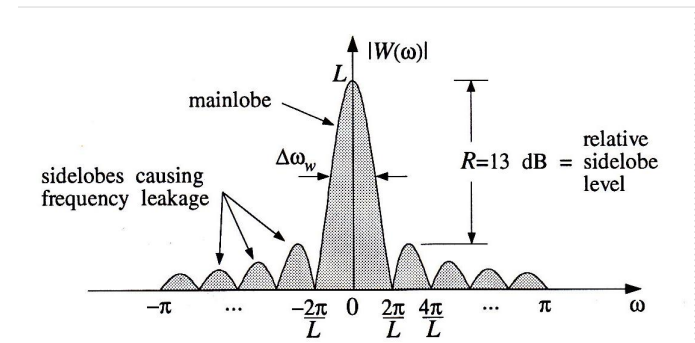
Reduces sidelobes

Reduces leakage

Narrows the main lobe

Some windowing function:

- Welch
- Hamming
- Hanning
- Kaiser
- ...



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Allan variance (AV) - D. Allan 1966

Alternative measure of variability

= “variations of sample averages of different length τ ”

$$\bar{X}_k(\tau) = \frac{1}{\tau} \sum_{j=0}^{\tau-1} X_{k-j} \quad \sigma_{\bar{X}}^2(\tau) = \frac{1}{2} \mathbf{E} \left[\left(\bar{X}_k(\tau) - \bar{X}_{k-\tau}(\tau) \right)^2 \right]$$

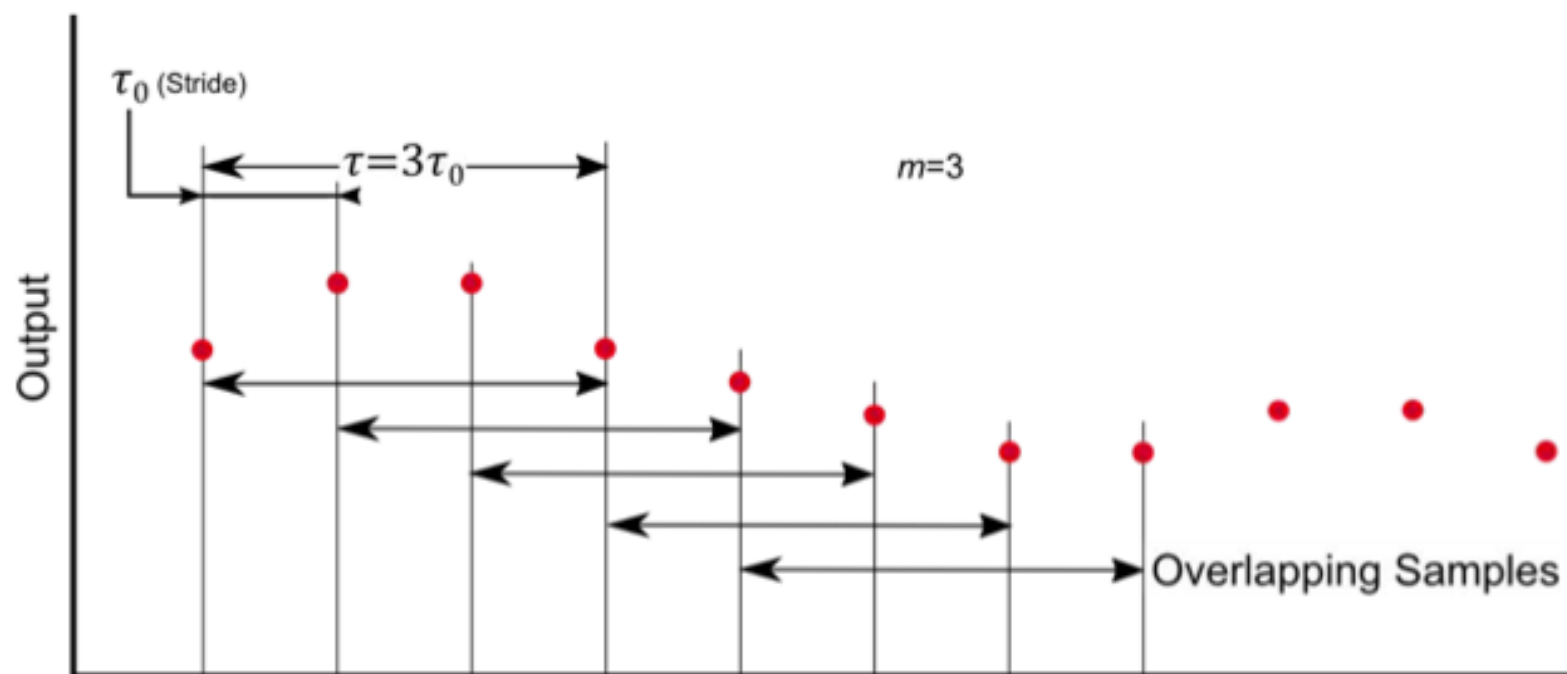
Greenhall (1991) estimator of AV

$$\hat{\sigma}_{\bar{x}}^2(\tau) = \frac{1}{2(N - 2\tau + 1)} \sum_{k=2\tau}^N \left(\bar{x}_k(\tau) - \bar{x}_{k-\tau}(\tau) \right)^2$$

$x_k : k = 1, \dots, N$

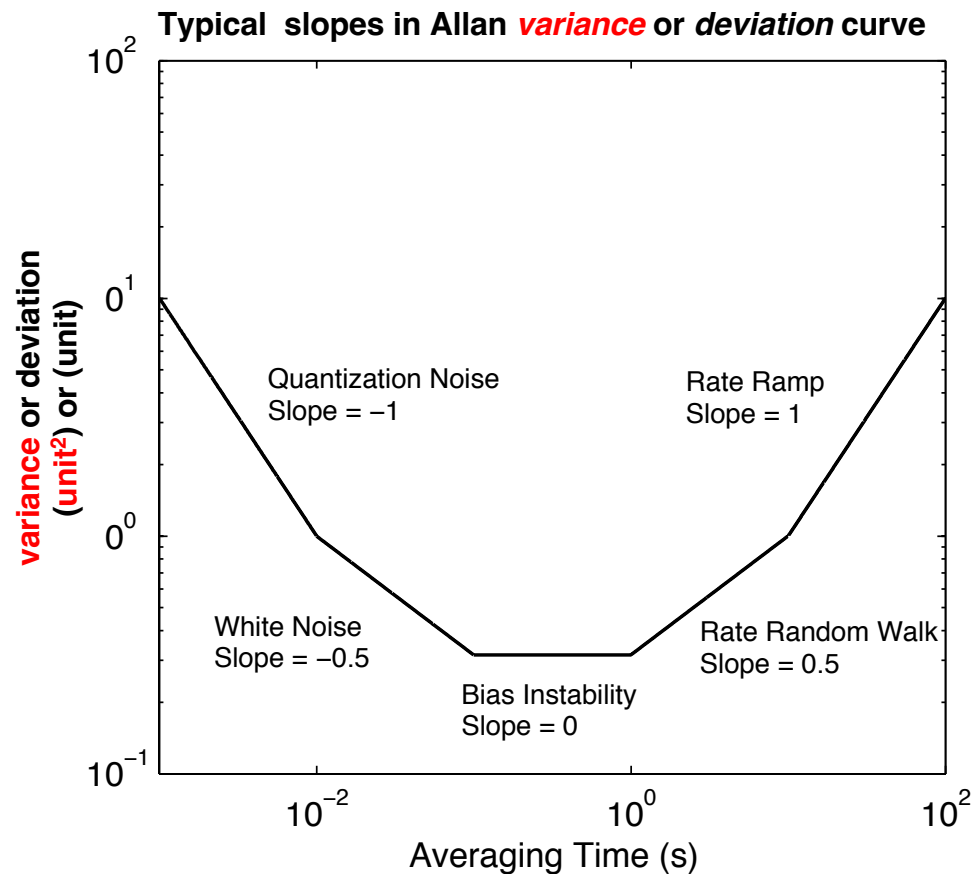
Allan variance – with sample overlap

Concept of AV with overlapping samples



source: Freescale Semiconductor, Application note #AN5087, Rev. 0,2/2015

Process identification in AV plots

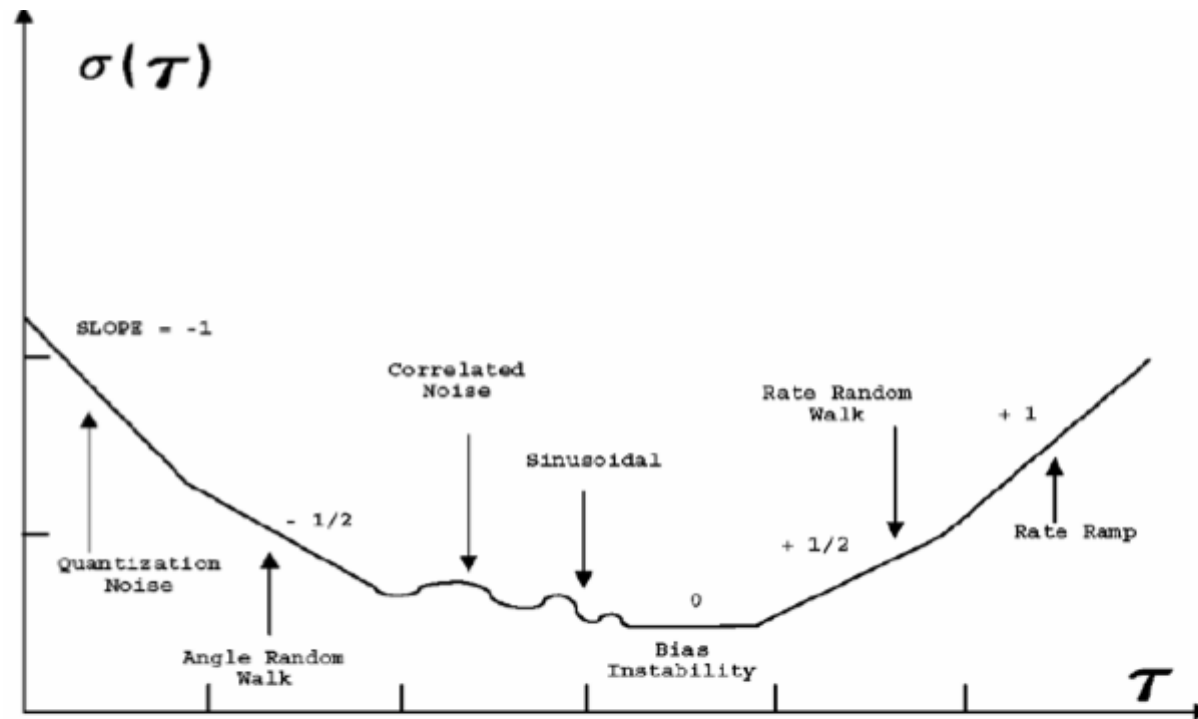


Remarks:

- Forward mapping:
 - Easy
 - IEEE standard
- Inverse:
 - not always
 - not a standard
 - Use GMWM ...

Allan deviation – process signature

Complex error structure -> parameter estimation difficult



AV of a "static" gyro, source: scientific publication

Transformation into a “non-latent” model (e.g. ARMA)

- Does not work in general
- Difficult to “inverse”

“Graphical” / lin. regression method

- Limited to a few possible models
- Not consistent in general and “inefficient”

Maximum Likelihood Estimation (MLE) / EM algorithm

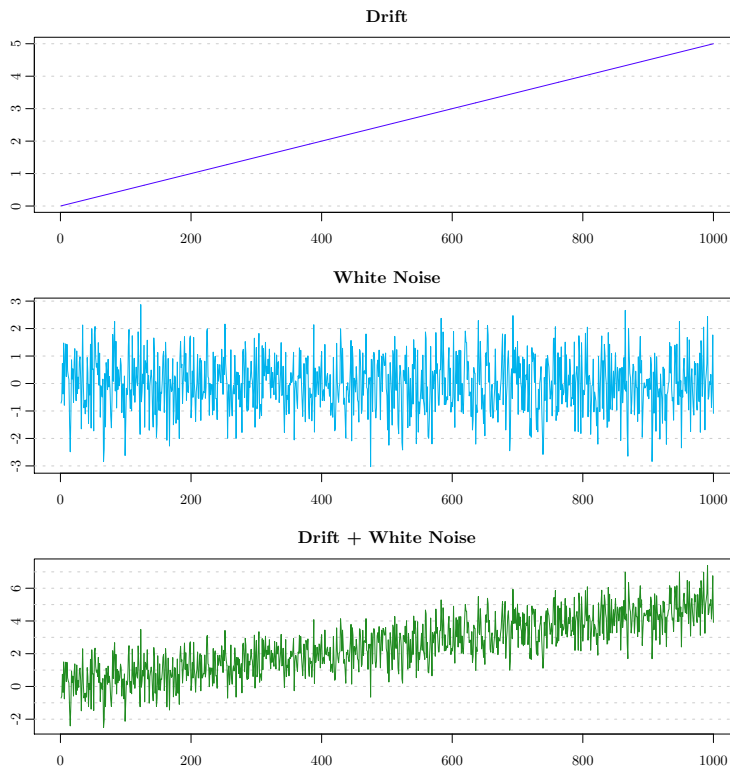
- Computationally intensive, laborious for new models
- Diverges with “complex” models

General Method of Wavelet Moment (GMWM)

- Rigorous, precise and efficient (*from students in this course*)
- Released as a freely available package in “R” (v1. 2015)
- Released as a GUI online tool (v1. 2018)
- <http://ggmwm.smac-group.com/>

Simple example 1

- Stochastic process: WN + ramp/drift



- Simple linear regression model:

$$y_t = \omega t + \varepsilon_t$$

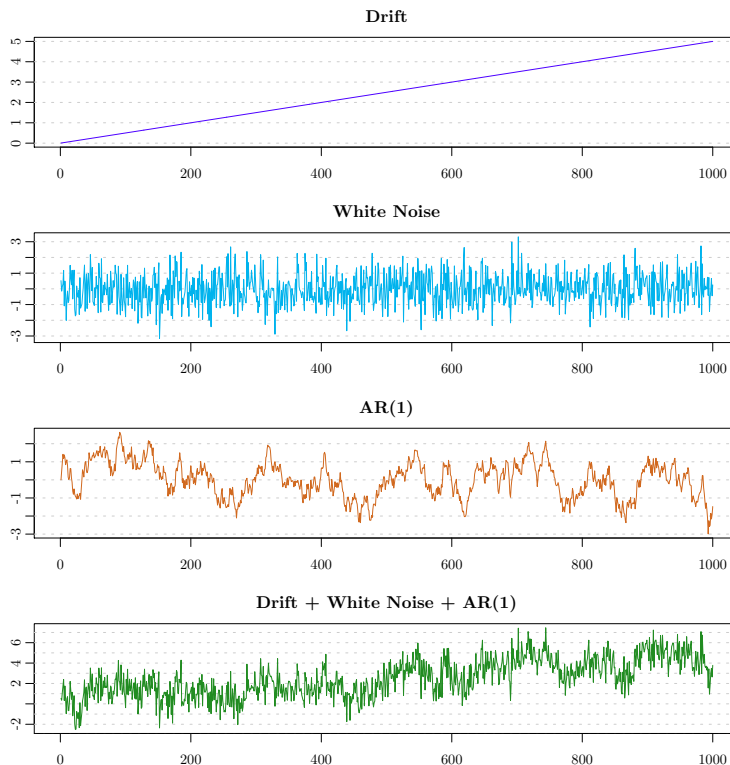
$$\varepsilon_t \sim N(0, \sigma^2)$$

- MLE is perfectly fine

- What about if we add AR(1) (e.g. GM) process?

Simple example 2

□ Stochastic process: WN + ramp/drift + AR1(GM1)



- Not a linear regression model but **state-space model**
- Computing the likelihood is not an easy task (e.g. Kalman filter)
- Practice: MLE (in fact EM-KF) fails

Fact: AV is a special case of the wavelet variance (WV)

- $AV = 2 * WV$ (for Haar wavelet)

Match Wavelet Variances

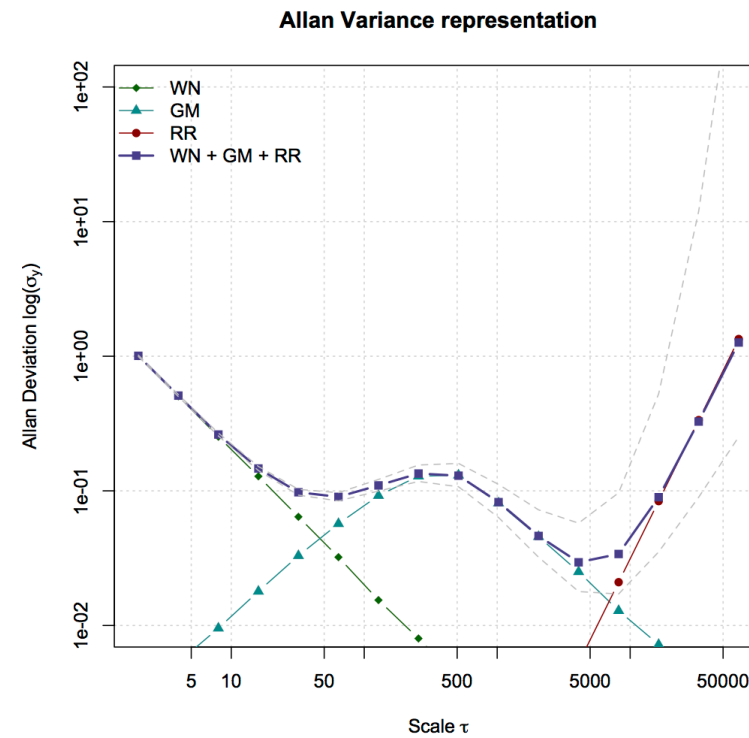
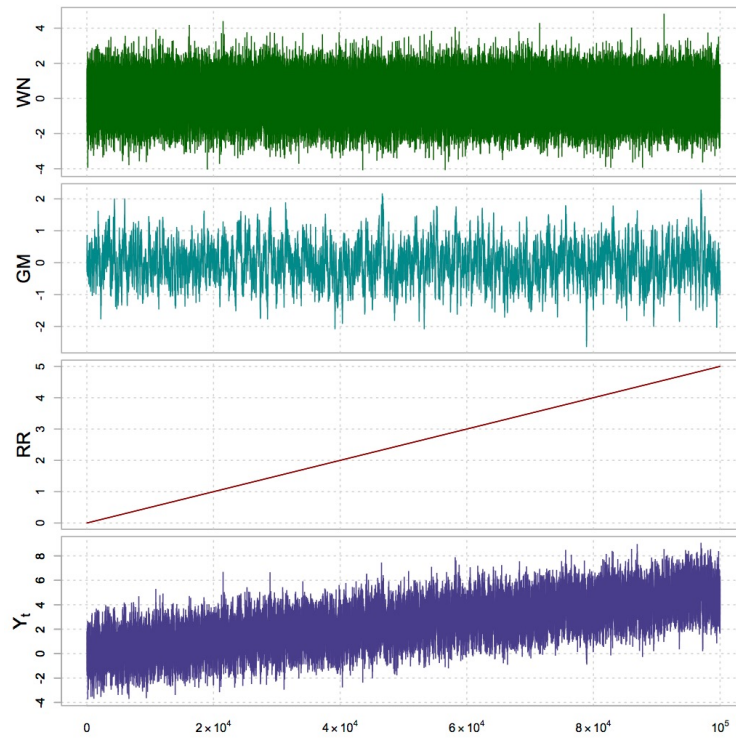
- Exploit the known relation between a model θ and its “WV signature” $v(\theta)$
- Inverse the mapping by minimizing the discrepancy between WV from data: $\hat{v}$ and $v(\theta)$ from model:

Solution:

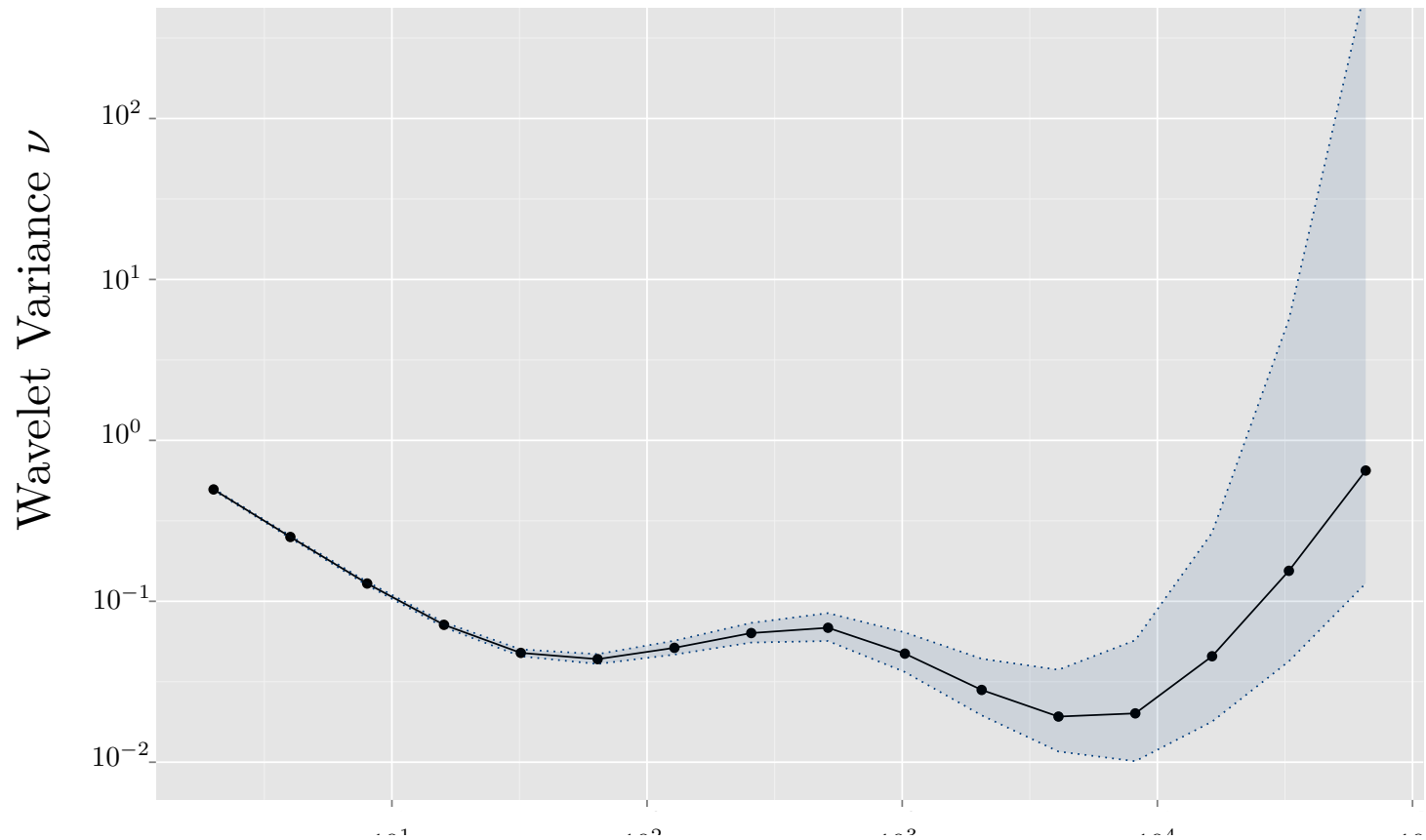
$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmin}} \left(\hat{v} - v(\theta) \right)^T \Omega \left(\hat{v} - v(\theta) \right)$$

Example

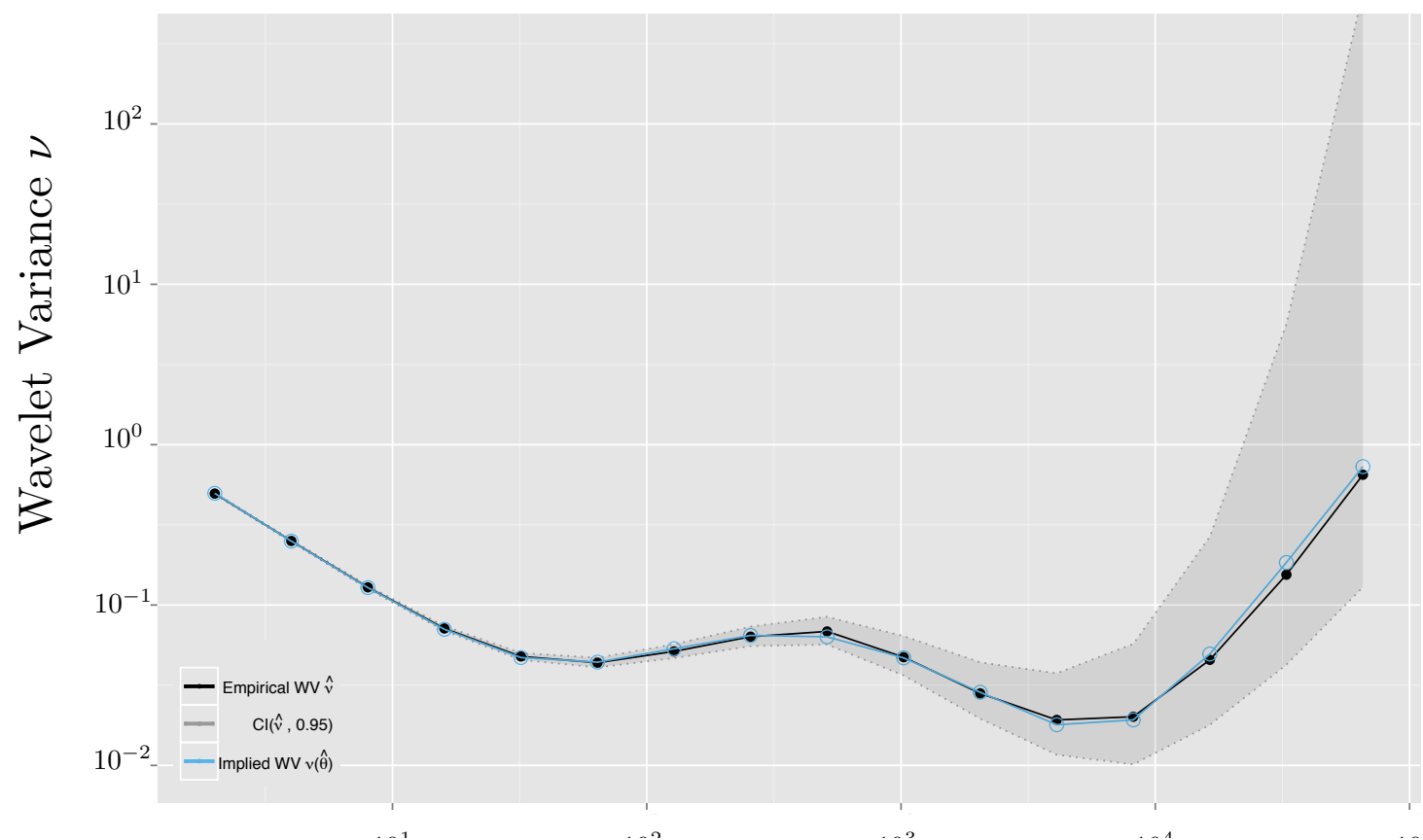
□ Composite stochastic process: WN + GM1 + RAMP



Empirical Wavelet Variance (= 2 x AV) of the example



GMWM estimation results



GMWM estimation results

Estimated parameters

	θ_0	$\hat{\theta}$	IC ($\theta_0, 0.95$)
σ_{WN}^2	1.00	1.00	(0.99; 1.01)
σ_{GM}^2	0.60	0.58	(0.55; 0.61)
β	10^{-2}	$1.07 \cdot 10^{-2}$	$(0.99 \cdot 10^{-2}; 1.12 \cdot 10^{-2})$
ω	$5 \cdot 10^{-5}$	$4.87 \cdot 10^{-5}$	$(4.67 \cdot 10^{-5}; 5.07 \cdot 10^{-5})$

Goodness of fit test:

- p-value ~ 1 (i.e. we cannot reject the proposed model)